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Identifying Nonlinearity in Financial Time Series

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Abstract

The nonlinearity of financial time series requires the application of different tests to identify it, some of them coming out from outside financial field. Different outlooks of such tests appear scattered in the literature. We give some general essential considerations on nonlinear systems and then we describe three tests: the rescaled range (R/S) analysis for the estimation of the Hurst exponent, the Brock, Dechert and Scheinkman (BDS) statistics, and the estimation of the largest Lyapunov exponent. The aim of the present work is to present a review of the latest versions of these tests and to compare them by applying to the S&P 500 index.

1. Introduction

In a recent paper, Abecasis, Lapenta and Pedreira [1] pointed out that one of the most challenging approaches in the analysis of modern time series is the area of financial time series. This represents the first stage in our research plan on the study of financial time series. The present work deals with our second stage on the subject under consideration that involves the identification of nonlinearity in financial time series. At present it is a well-established fact that they are non-random and nonlinear. Of course, it is an average behavior since the series are nonstationary. Therefore one is led to know not only the kind of the time series but also its preprocessing in order to feed the series to the computational intelligent artifact.

Versions of nonlinearity tests found in the literature are not necessarily connected with financial time series. As a matter of fact, not all the publications are lightly attainable to most practitioners, applied researchers as well as novices. This is due to the fact that the subject under consideration is not easily manageable since it implies knowledge of a group of disciplines that not everyone is required to dominate. For this reason, we think that the present work on the three tests: the rescaled range (R/S) analysis for the estimation of the Hurst exponent, the Brock, Dechert and Scheinkman (BDS) statistics, and the estimation of the largest Lyapunov exponent together with their software versions, would be a useful and practical tool for solving the problem of the characterization of financial time series involving the identification of their nonlinearities. Different outlooks of these tests appear scattered in the literature. Peters [2,3] and Brock, Hsieh and LeBaron [4] present a comprehensive overview of nonlinear systems especially devoted to financial ones. Here, we only give some details to understand the three complementary tests for nonlinearity to be dealt with in the following sections. Chaos theory, originally introduced in the study of thermodynamical nonequilibrium systems, has entered in the study of financial markets. The theory predicts a statistical order in complicated systems, which appear disordered; it serves also to treat the interdependence of variables established by nonlinearity. Another feature of nonlinear systems is that there is a Strong Dependence on Initial Conditions (SDIC) that prevents making forecasts unless they are very short term, how short it depends on the system. SDIC implies that small changes in the state of a system will enlarge at an exponential rate. Therefore, the long-term predictions of chaotic systems are vain, independently of the way the system prediction is implemented. SDIC is quantified with Lyapunov exponents. Positivity of the largest Lyapunov exponent is one way to capture the notion of SDIC that is the hallmark of chaos. A system may possess any number of Lyapunov exponents, but to confirm chaos one only needs to confirm that the largest Lyapunov exponent is positive. Consequently, estimation of its value is of extreme importance. In order to describe nonlinear systems it has been found that the appropriate geometry is *fractal geometry*. The name comes from the fact that it deals with figures with a fractional dimension having properties to be later explained. The essential characteristic of a fractal curve is that at every place whichever the chosen scale, the observed part is similar to the whole. This selfsimilarity is characteristic of fractals. Time series in capital markets are fractals in time; the selfsimilarity is observed comparing price series in annual, monthly, and daily scales. The selfsimilarity reflects an internal correlation, which shows up as a long memory, which is infinite for mathematical fractals but finite for real, stochastic ones. The BDS test distinguishes between random systems from deterministic chaos and from nonlinear stochastic systems under the condition to remove any linear dependence in the data before one can use BDS to test nonlinearity. However, it cannot detect nonlinear deterministic systems (chaos) from nonlinear stochastic systems. The existence of an underlying chaos can be demonstrated by *R/*S analysis or by the determination of the largest Lyapunov exponent. Both reveal a characteristic feature of chaos: the existence of statistical cycles. If the cycle length does not depend on either sample size or the frequency of the data, one can be sure that the system is chaotic.

Section 2 is devoted to the rescaled analysis and the estimation of the Hurst exponent. Section 3 deals with the BDS test that identifies all types of nonlinearities. Finally, Section 4 describes the calculation of the largest Lyapunov exponent that serves to identify chaos. The three tests are applied to monthly averaged values of the S&P 500 index from February 1974 up to February 1999.

2. R/S analysis and the Hurst exponent

Hurst [Peters 3, p.62] found the following relation for a series of *n* hydrological observations

$$\left(R/S\right)_n = cn^H \,, \qquad (2.1)$$

where R_n is the range of values for index *n*, *S* is the local standard deviation, *c* is a constant, and *H* is at present called the *Hurst exponent*. The *R/S* value of eq. (2.1) is referred to as the *rescaled range* because it has zero mean and is expressed in terms of the local standard deviation. According to statistics *H* must be equal to 0.50 if the series is a random one (not necessarily of Gaussian structure). Many financial time series have a Hurst exponent greater than 0.50. This means that the series has a long-range memory according to which each data remembers the previous one; what happens today influences the future. When *0.50 < H £ 1.00* the series is *persistent*; if it is up (down) in some period it is most probable that it continues up (down) in the next period. The opposite behavior appears when *0 £ H < 0.50*; these series are *antipersistent* and show strong oscillations. In some occasions these series are called *mean reverting* but this name is inappropriate because in mean reversion the mean remains constant, something that cannot be assured for antipersistent series. Hurst exponent gives the probability that if a change was positive the following would also be positive. As every point is not equally probable, the fractal dimension of the probability distribution is not two (as in a random system) but rather a number different from two; less than two for persistent systems and greater than two for antipersistent series. Mandelbrot [Peters 3, p.66] has shown that the fractal dimension of a time series is equal to the inverse of *H.* A significant result of the rescaled analysis is that it shows the existence of a statistical cycle characteristic of each time series, determined by the fact that for some *n* the eq. (2.1) ceases to hold. The existence of these cycles indicates that the initial conditions are forgotten.

In the following paragraphs, we present an algorithm to estimate the Hurst exponent. Consider a time series of N raw data a_i converted into a series of logarithmic data b_i. Take AR (1) residuals of these data to remove linearities, since there are autoregressive processes that can cause short-term correlations. The process is expressed as

$$y_t = b_t - \left(rb_{t-1} + m_t\right), \qquad (2.2)$$

where y_t is the transformed data at time *t*, $r \leq 1$ is a constant, and m_t is a number of an *Independently and Identically Distribution* (IID). Divide the whole set in *J* continuous and nonoverlapping subsets of length *n*, such that $J \bullet n = N$. The values of *n* include the beginning and ending points of the time series; and *N/n* is an integer value. Each element in each subset J_j with *j = 1,2, ...J*, is labeled $y_{k,j}$ with *k = 1,2,..., n*. Calculate the mean for each J_j,

$$\bar{y}_j = \frac{1}{n}\sum_{k=1}^{n} y_{k,j}\,. \qquad (2.3)$$

Calculate the standard deviation for each J_j,

$$S_j = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}\left(y_{k,j} - \bar{y}_j\right)^2}\,. \qquad (2.4)$$

Rescale the data by subtracting the sample mean for each J_j,

$$x_j = \sum_{k=1}^{n}\left(y_{k,j} - \bar{y}_j\right) \qquad (2.5)$$

Calculate the adjusted range for each J_j,

$$R_j = max(x_{k,j}) - min(x_{k,j}). \qquad (2.6)$$

Normalize this range dividing it by its standard deviation

$$(R/S)_j = R_j / S_j\,. \qquad (2.7)$$

Average the last values in order to obtain the rescaled range for length *n*,

$$(R/S)_n = \frac{1}{J}\sum_{j=1}^{J}(R/S)_j\,. \qquad (2.8)$$

Increase the length *n* a higher value. Repeat calculations from eqs. (2.3) to (2.8) inclusive, until *n = N/2*. Approximate the Hurst exponent *H* by plotting the *log (n)* (abscissas) versus $log(R/S)_n$ (ordinates), and performing an ordinary least squares regression on the first variable as the independent one and the second variable as the dependent one. From eq. (2.1), the slope of the equation – the estimation of the Hurst exponent H - is

$$log\left(R/S\right)_n \cong H\,log(n)\,, \qquad (2.9)$$

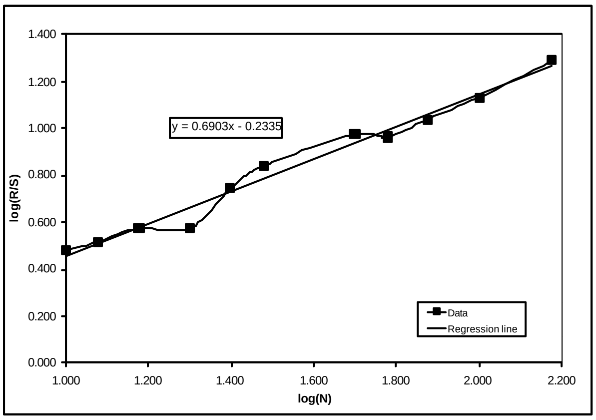

FIGURE 1: R/S analysis for S&P 500 logarithmic data monthly averaged from February 1974 - February 1999. Estimated *H* = 0.690.

3. The BDS test

Some previous considerations in the theory of nonlinear systems are necessary to understand the BDS test. A convenient and very useful form to visualize a system (linear or not) is to consider its *phase space*, which has as many dimensions as variables determine the system. A point in the phase space represents a possible state of the system, hence the alternative name *state space*. The region of space where solutions tend to lie receives the name of *attractor*. The orbits in phase space of a chaotic system have a diverging part followed by a folding in a non-periodic manner so that the system remains in a limited region called *strange attractor*. The problem of dimensionality of the space phase is not of interest for the present purpose. It suffices to know that the reconstructed phase space of financial time series have usually few dimensions (low dimensional chaos).

The economists Brock, Dechert, and Scheinkman [5] developed a test now called BDS test, that is a powerful tool to distinguish a random from a non-random system - see also Brock, Dechert, Scheinkman and LeBaron [6] and Brock and Potter [7]-. Essentially, it finds a nonlinear dependence but cannot distinguish between a nonlinear stochastic system and a deterministic chaotic one. Another test is necessary in this case (see Section 4).

The BDS test essentially consists in using the time delay method to build an m-dimensional space from the original time series of length N. Then, a correlation function $C_{m,N}$ (ε) is calculated

$$C_{m,N}(e) = \sum_{t<s} I_e(x_t^m, x_s^m)\left[2/T_m(T_m-1)\right], \qquad (3.1)$$

where

$$T_m = N - m + 1, \qquad (3.2)$$

and $I_e(x_t^m, x_s^m)$ is an indicator function that equals 1 if $\| x_t^m - x_s^m \| < \varepsilon$, and equals 0 otherwise; || || is the sup norm which means that it is equal to the largest absolute values of the m components of the involved vector. This function measures the fraction of pairs of points (x_t^m, x_s^m) that are within a distance ε of each other. Under the null hypothesis that the series is an IID, the following relation holds, with 100% probability, for fixed m and ε,

$$C_{m,N}(e) \rightarrow C(e)^m \text{ with probability 1, as } N \rightarrow \infty, \qquad (3.3)$$

where

$$C(e) \approx C_{1,N}(e) . \qquad (3.4)$$

This is the typical scaling feature of random process: the correlation function fills the space whatever the dimension is placed in. In the book by Brock, Hsieh and LeBaron [4] - which we will abbreviate BHL- the authors showed that

$$C_{m,N}(e) - C(e)^m \qquad (3.5)$$

is normally distributed with zero mean and variance σ^2_m (ε) defined in eq. (2.1.3) of BHL. The BDS statistics that follows (called BDS/SAD in the recipe of this section)

$$W_{m,N}(e) = T_m^{1/2}\left[C_{m,N}(e) - C_{1,N}(e)^m\right] / s_{m,N}(e), \qquad (3.6)$$

has therefore a limiting standard normal distribution under the null hypothesis that the system is IID.

There is one important point: the BDS test finds linear as well as nonlinear dependence in the data. Therefore, it is necessary to remove any linear dependence before one can use BDS to test nonlinearity. One of the simplest ways is to consider a first order autoregressive model AR (1), as we have done in section 2. The actual performance of the test requires that some points be taken into account. Consider, for instance, the value of ε relative to series standard deviations; if it is too large there will be too many points whereas if it is too small there will be not enough points to capture the statistical structure. Peters [2] suggests, following some examples, to take ε equal to half the standard deviation of the data set. In addition, an embedding dimension *m* must be chosen so that the points in the reconstructed phase space are neither too sparse nor too crowded. BHL recommended a choice of m between 2 and 5 for small data sets (200 to 500 observations), and up to 10 for large data sets (at least, 2000 observations). Unfortunately, as stated above, the test cannot distinguish between chaotic and stochastic behavior. To become a really powerful tool it must be used in conjunction with other tests described in the present paper. BHL performed Monte Carlo calculations. So, in Appendix C of their book they present quantiles of BDS statistics for normal random variables for 100, 250 and 500 observations for ε = 0.5 δ and ε = δ, and m = 2, 3, 4, 5, and 10. The last column labeled N (0,1) indicates the deviations from normal distributions quantiles for zero mean and unit variance.

The rest of this section is devoted to the description of a software -the most appropriate in our opinion- that implements the BDS test, written by W. Davis Dechert [8]; also see Dechert [9].

Select an Input File. Before the use of the BDS to test for nonlinearity, any linear dependencies in the data to be analyzed should be removed.

Descriptive Statistics. A typical screen looks like the following:

Input File: SP500.TXT N = 300

Data Description

Initial Observation: 1
Number of Observations: 300
Minimum = 2.926E+0001
Maximum = 6.544E+0002
Spread = 6.251E+0002
Average = 1.610E+0002
Std Dev = 1.361E+0002
SD / Spread = 2.176E-0001

Regression on t

Coefficient of t = 1.3664E+0000
T statistic = 2.0419E-0002
R squared = 7.5891E-0001

Choose Epsilon. The value of epsilon chosen will be multiplied by the spread of the data set. To select the value of ε, choose a fraction of the spread of the data shown in the above dialog box. For example, to pick a ε value that will correspond to half the standard deviation, choose one half of the value of *SD/spread* that is reported in that dialog box.

Choose the Embedding Dimension m.

Calculate Cm & BDS Statistics. The screen output consists of *C, K, C1* and *Cm* which are the integer number of observations in each category specified by *N*, *m*, and ε. The BDS statistics, its standard deviation and the ratio of the two are also displayed. A screen looks similar to this:

Input File: SP500.TXT N = 300
Epsilon = 1.088E-0001 m = 4
C = 34780
K = 4844566
C1 = 17240
Cm = 15281
αm = 4.7631E-0001
BDS = 5.6149E+0000
SD = 3.5095E-0002
BDS/SD = 1.5999E+0002

Look at the Results. To look at the results after they were stored on a file, retrieve the utility program LIST.COM by typing LIST filename.ext at the DOS prompt. The following is a sample of the information that is written to the file.

FILENAME: SP500.TXT
Initial Obs : 1 Num Obs : N = 300 SD/Spread = 2.1765E-0001

Epsilon	m	C1	Cm	BDS	SD	BDS/SD
0.1088	5	17240	14899	5.726E+00	2.034E-02	2.814E+02
0.1088	4	17240	15281	5.614E+00	3.509E-02	1.599E+02
0.1088	3	17240	15719	5.151E+00	5.303E-02	9.712E+01
0.1088	2	17240	16304	3.773E+00	6.017E-02	6.269E+01

4. Lyapunov exponents from a time series

As stated before, one of the important characteristics of chaotic systems is a sensitive dependence on initial conditions. The susceptibility of a system to this dependence can be measured with certain numbers ë called Lyapunov exponents. There is one λ_i for each dimension *i* of the phase space and are defined by the expression

$$l_i = \lim_{t\to\infty}(1/t)\log_2\left(p_i(t)/p_i(0)\right), \qquad (4.1)$$

where p_i is the coordinate of a point in dimension *i*.

Consider an attractor in three dimensions that has one exponent zero and two negative ones: two dimensions converge one on the other and one in which there is no change in the relative position of points. In the most interesting case for this work, strange attractors in three dimensions have one zero exponent, one negative and one positive. The positive Lyapunov exponent measures the dependency on initial conditions that is the trend of small changes in the initial conditions to strongly alter forecasting. The negative exponent makes points tending to diverge be maintained within the region of the attractor. The existence of a positive Lyapunov exponent for a system demonstrates that it is nonlinear, a test independent of the determination of the Hurst exponent in the R/S analysis. Wolf, Swift, Swinney and Vastano [10] have developed an algorithm to calculated the largest Lyapunov exponent λ_1 starting from experimental data. Its formal expression – according to Wolf [11]- is

$$l_1 = (1/t)\sum_{j=1}^{m}\log_2\left(L'(t_{j+1})/L(t_j)\right), \qquad (4.2)$$

where $L(t_j)$ is the distance between two starting points, $L'(t_{j+1})$ is the distance between them at a later time, and *j* is the sample number. This equation is equivalent to eq. (4.1) when one deals with an infinite amount of noisy-free data. However, the practical conditions are quite different. Fortunately (see below), the rules of thumb given by Wolf allow dealing with experimental data. The method requires a lot of numerical experimentation, mainly in case of failure, in order to be sure that the result is not due to a bad selection of parameters or to insufficient data. It must be pointed out that the method yields a largest average Lyapunov exponent, since the starting time series includes all possible phases of the market, including random walk and chaos. It is important to note that the inverse of the largest Lyapunov exponent gives the length of the statistical cycle that appeared in R/S analysis. The reader is referred to Section 5 of Wolf and Bessoir [12] that contains the most modern thoughts on the Lyapunov exponents. Wolf et al. [10] have developed a method that allows the estimation of non-negative Lyapunov exponents from an experimental time series. The Home Page of Alan Wolf [13] offers newer and more efficient FORTRAN and C codes of the program. This document is the first official program documentation for the so-called *BASGEN* (dataBASeGENerator) and *FET* (Fixed Evolution Time) programs. Both make use of the method of *phase space reconstruction* also known as *time delay reconstruction*. The method builds a *ndim*-dimensional orbit out of a time series once the user selects two parameters: the embedding dimension *ndim* and the time delay *tau*. The following paragraphs present an excerpt of the programs detailed by Wolf [13].

BASGEN reads the original time series from the ASCII file data - with one value per line-, creates a database file, and a new time series. The values for the input parameters are: *ASCII data file = 1:* For the first run. *Number of data points (£ 32000)*: corresponding to a fractal dimension *d* = 4, approximately. *Time delay [tau] (samples):* roughly one third of the number of points in the mean period of the motion. *Embedding dimension [ndim] (£8):* A good value to start with is four. *Grid resolution [ires] (maxbox = 6000):* Typical values for *ires* are in the range of 6 to 12. The output values from BASGEN are the *number of boxes allocated*, *created*, and *non-empty*.

FET estimates the dominant (largest positive) Lyapunov exponent in a time series. FET creates a multidimensional phase space orbit from a one-dimensional time series by phase space reconstruction. Using the database created by BASGEN, FET locates a pair of points that are very close to each other in the reconstructed phase space orbit. FET follows each of the points as they travel a short distance along the phase space orbit. One can compare the initial separation (ordinary Euclidean distance) of these points to their separation at the end of interval. The logarithm (base 2) of the ratio of final to initial separation of these points is a local estimate of orbital divergence. If the two points are still close together at the end of this interval, one evolves them a bit farther along the orbit, then computes the next local value of orbital divergence. On the other hand, if they are much farther apart, one keeps one of the points, and uses the database to find an appropriate replacement for the other point. One obtains the long-time average rate of divergence of nearby orbits by averaging the local rates of orbital divergence and dividing by the total travel time along the orbit. The values for its input parameters are: *Time-step (seconds or iterations):* It is the time between samples in the time series. The entrance one implies that the Lyapunov exponent has the units of bits per seconds. *Evolution time [evolve] (number of samples):* It should be kept small enough so that orbital divergence is monitored at least a few times per orbit, and sensitive dependence does not pull the points too far apart. *Minimum separation at replacement [dismin]:* Try zero to two percent range of time series values. *Maximum separation at replacement [dismax]*: Try 10% to 15% of range of time series values. *Maximum orientation error [thmax]:* Try 30 degrees. The output value of FET is the Lyapunov exponent.

Table 1: Present input and output values to estimate the largest Lyapunov exponent

PROGRAM	ITEM	VALUE
Input values to BASGEN	ASCII data file	1
	Number of data points	300
	Time delay [tau] (samples)	6
	Embedding dimension [ndim]	4
	Grid resolution [ires]	7
Output values from BASGEN	# boxes allocated	6000
	# boxes created	74
	# boxes non-empty	41
Input values to FET	Time-step (seconds or iterations)	0.9
	Evolution time [evolve] (number of samples)	6
	Minimum separation at replacement [dismin]	0.0001
	Maximum separation for replacement [dismax]	31.26
	Maximum orientation error [thmax]	30.

The present results are summarized in Table I corresponding to the S&P 500 data from February 1974 to February 1999. The Lyapunov exponent converged to an averaged value of 0.0328 bits/month. That is, one loses all predictive power after 1/0.0328 or 30 months. Since the estimated exponent is near zero the system is exhibiting some orbital stability or periodicity.

5. Conditional mean and conditional variance

Until recently, for the purpose of detecting nonlinearity in financial time series very few works have considered the problem of constructing joint tests that would help considering the nonlinear dynamics that are present both in the conditional mean and in the conditional variance of financial series. Gilles Dufrenot and Laurent Mathieu [14] applied a test to a series of high frequency exchange rates and proposed a class of models that allows the presence of non-linearity in both the conditional mean and conditional variance. The authors proposed a combination of exponential autoregressive models (EXPAR) and autoregressive conditional heteroskedasticity models (ARCH).

6. Concluding remarks

The aim of this work is to present a review of the ability as well as the compatibility of three tests for identifying nonlinearity in financial time series. They are: Hurst exponent, BDS statistics, and Lyapunov exponent. With this purpose we used a single data set constituted by 300 monthly averaged values of the S&P 500 index from February 1974 to February 1999. We chose this commonly used set to concentrate the attention of the users almost exclusively on the tests and their corresponding softwares. For the R/S analysis we smoothed the raw data by taking their logarithms. No such smoothing is necessary for the BDS test or for the calculation of the largest Lyapunov exponent, since the phase space reconstruction and the subsequent selection of points provides a statistical smoothing.

In summarizing sections 2-4 we list the present results: H = 0.690, BDS statistics = 159.99, and Lyapunov exponent = 0.0328. Since the Hurst exponent is greater than 0.50, the S&P 500 index exhibits the Hurst phenomena of persistence. The value of the BDS statistics implies that the null hypothesis of the randomness of the system can be rejected with 99% of confidence. The largest Lyapunov exponent is greater than zero and therefore, sensitivity dependence on initial conditions exists and there is a strange attractor for the system.

Not all the three tests are strictly comparable. In a sense, Hurst and Lyapunov exponents measure the amount of nonlinearity in a time series. Moreover, the latter measures the rate of decay of forecast accuracy; in the present case it is approximately equal to 30 months. In our study both are compatible between them and between other results such as those by Peters [2, 3]. Therefore, one faces a deterministic chaotic system. BDS test measures essentially the degree of confidence in the presence of nonlinearity. But we wish to stress that it cannot distinguish between nonlinear deterministic and nonlinear stochastic systems. The values obtained here are higher than those tabulated by BHL in Appendixes C.2 and C.3 even for embedding dimensions higher than 10. Moreover, note that the embedding dimension in both BDS test and the estimation of the Lyapunov exponent are equal (m = 4). This value appears to be very reasonable because the fractal dimension of the system is 1/H = 1.45. This is only a tentative value since to really estimate it within a precise context it would be necessary a great number of data. We indicate it to have an idea of the value of the embedding dimension.

The paper of Dufrenot and Mathieu [14] opens a research plan with possible extensions to other nonlinear models and applications to other financial time series.

We think that the present work is of concern to the understanding of the important issue on nonlinearities identification in financial time series hoping that it will serve as a vademecum to practitioners, applied researchers as well as non-specialists.

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